



On almost small and almost large super-Vandermonde sets in $GF(q)$

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Abstract A set $T \subset GF(q)$, $q = p^h$ is a *super-Vandermonde set* if $\sum_{y \in T} y^k = 0$ for $0 < k < |T|$. We determine the structure of super-Vandermonde sets of size $p + 1$ (almost small) and size $q/p - 1$ (almost large).

Keywords Finite geometry · Finite fields · Vandermonde set

Mathematics Subject Classification 05E · 51E

1 Introduction

A *super-Vandermonde set* (short: an sV-set) in $GF(q)$, $q = p^h$, p a prime, is a set T of size $1 < t < q$ such that

$$\pi_k(T) := \sum_{y \in T} y^k = 0,$$

for $0 < k < t$. It follows from the non-singularity of the Vandermonde matrices $(y^k)_{y,k}$, $y \in T$ and $k \in [0, t)$ resp. $k \in (0, t]$ that $0 \notin T$ and that $\pi_t(T) \neq 0$ (in particular $p \nmid t$). The Newton identities relating the power sums $\pi_k(T)$ and the elementary symmetric polynomials

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$\sigma_k(T)$ imply that in the polynomial

$$f(Z) := \prod_{y \in T} (Z - y) = \sum (-1)^k \sigma_k(T) Z^{t-k},$$

the only possible nonzero coefficients are the constant term $(-1)^t \sigma_t$ and the coefficient of Z^{t-k} : $(-1)^k \sigma_k$ with $k \equiv 0 \pmod p$. The Newton-identities are given by:

$$k\sigma_k = \sum_{m=1}^k (-1)^{m-1} \pi_m \sigma_{k-m},$$

and we see that indeed $\sigma_k = 0$ if k is not divisible by p (and less than t).

In terms of the inverses of the elements in T , we get that being sV is equivalent to

$$\phi(Y) := \prod_{y \in T} (Y - y^{-1}) = Y^t + g(Y),$$

with g a p -th power.

The underlying notion of Vandermonde set was introduced by Gács and Weiner in [1]. They appear at several places in the investigation of special point sets in finite projective planes. More about this, as well as many examples, can be found in Chapter 1 of the thesis of Takáts [2], or in her paper [3] with Péter Sziklai, which also classifies small and large sV-sets. Here small means $t < p$, and small sV-sets are cosets of multiplicative subgroups of $GF(q)^*$: in this case the polynomial g is constant, so

$$\phi(Y) = \prod_{y \in T} (Y - y^{-1}) = Y^t - c,$$

where $t \mid q - 1$ and c is a t -th power, so that T is a coset of the group of t -th roots of unity.

By large we mean $t > q/p$ and again we get cosets of multiplicative subgroups, corresponding to the case that $g = -c$ is constant. The proof in this case is much more involved, but in the final section we will give a simpler proof.

2 Super-Vandermonde sets of size $p + 1$

If T is an sV-set of size $p + 1$, then the polynomial $\prod_{y \in T} (Z - y)$ is of the form $f(Z) = Z^{p+1} + aZ + b$, so our problem is to classify the polynomials of this form that are fully reducible over $GF(q)$. Notice that two different polynomials of this form have a gcd of degree at most one, so that two elements of $GF(q)$ are contained in at most one sV-set of size $p + 1$. We will see in fact that two elements are contained in an sV-set of this size precisely when they have the same $GF(p)$ -norm. We will prove in the next theorem that they can all be obtained from 2-dimensional $GF(p)$ -vector subspaces of $GF(q)$.

Theorem 2.1 *Let T be an sV-set in $GF(q)$, $q = p^h$, p prime, of size $p + 1$. Then there exists $\alpha \in GF(q)^*$ such that*

$$T = \{\alpha x_1^{p-1}, \dots, \alpha x_{p+1}^{p-1}\}, \quad (1)$$

where $\{x_1, \dots, x_{p+1}\}$ represent the 1-dimensional subspaces of a 2-dimensional $GF(p)$ -vector subspace of $GF(q)$.

Conversely, every 2-dimensional $GF(p)$ -vector subspace of $GF(q)$ defines a family of $q - 1$ sV-sets of type (1). In particular, the elements of an sV-set of size $p + 1$ have the same norm over $GF(p)$.

Proof We first observe that if $T = \{y_1, \dots, y_t\}$ is an sV-set, then for each $\gamma \in GF(q)^*$, the set $\gamma T = \{\gamma y_1, \dots, \gamma y_t\}$ is an sV-set as well (and of the same size of course). We first show that 2-dimensional subspaces give rise to sV-sets. Let U be a 2-dimensional $GF(p)$ -vector subspace of $GF(q)$, then U is the set of zeros of a polynomial of the form

$$X^{p^2} + aX^p + bX, \quad (2)$$

for some $a, b \in GF(q)$. If x_1 and x_2 are two nonzero roots of (2) which are not proportional over $GF(p)$, then x_1^{p-1} and x_2^{p-1} are two different roots of the polynomial $Z^{p+1} + aZ + b$, which turns out to be fully reducible over $GF(q)$. It follows that for each $\alpha \in GF(q)^*$

$$\alpha T = \{\alpha x^{p-1} : x \text{ is a nonzero root of (2)}\}$$

is an sV-set of size $p + 1$.

On the other hand let $T = \{y_1, \dots, y_{p+1}\}$ be an sV-set of size $p + 1$ and let

$$f(Z) = Z^{p+1} + aZ + b \quad (3)$$

be the associated polynomial. Then, there exist $y_i, y_j \in T$, with the same $GF(p)$ -norm δ . Let α be an element of $GF(q)^*$ with norm $N(\alpha) = \delta$ and set $z_k := y_k/\alpha$, for $k \in \{1, \dots, p+1\}$. Then

$$\frac{1}{\alpha}T := \{z_1, \dots, z_{p+1}\},$$

is an sV-set of size $p + 1$ with $N(z_i) = N(z_j) = 1$ and its associated polynomial is

$$Z^{p+1} + \frac{a}{\alpha^p}Z + \frac{b}{\alpha^{p+1}}.$$

Denoting by x_i and x_j the elements of $GF(q)^*$ such that $z_i = x_i^{p-1}$ and $z_j = x_j^{p-1}$, then x_i and x_j are independent over $GF(p)$ and so $U := \langle x_i, x_j \rangle$ is a 2-dimensional $GF(p)$ -vector subspace of $GF(q)$, whose elements are the zeros of the polynomial

$$X^{p^2} + \frac{a}{\alpha^p}X^p + \frac{b}{\alpha^{p+1}}X.$$

It follows that the elements of $\frac{1}{\alpha}T$ are of the form x^{p-1} . This completes the proof. \square

3 Super-Vandermonde sets of size $q/p - 1$

Consider the polynomial $\text{Tr}_{q \rightarrow p}(aZ) = aZ + a^pZ^p + \dots + a^{p^{h-1}}Z^{p^{h-1}}$, the trace from $GF(q)$ to $GF(p)$. It is clearly fully reducible over $GF(q)$, and we see that the nonzero roots form an sV-set of size $q/p - 1$. The aim of this section is to prove the converse:

Proposition 3.1 *Let T be an sV-set in $GF(q)$, of size $q/p - 1$, ($q = p^h$) then*

$$\prod_{y \in T} (Z - y) = (a_{h-1}Z)^{-1} \text{Tr}_{q \rightarrow p}(aZ)$$

for some $a \in GF(q)^*$.

Proof Consider as before the polynomial

$$\phi(Y) = \prod_{y \in T} (Y - y^{-1}) = Y^{q/p-1} + g(Y),$$

where g is a p -th power. Let T_a be the sV-set corresponding to the hyperplane $\text{Tr}(aZ) = 0$ with

$$\phi_a(Y) := \prod_{y \in T_a} (Y - y^{-1}) = Y^{q/p-1} + g_a(Y).$$

The greatest common divisor of ϕ and ϕ_a divides $(g(Y) - g_a(Y))^{1/p}$ of degree at most $q/p^2 - 1$. So we find that T has at most $q/p^2 - 1$ points in every hyperplane, unless it coincides with it. Since the average size of the intersection of T with a hyperplane equals

$$\frac{q/p-1}{q-1} \cdot \left(\frac{q}{p} - 1\right) > \frac{q}{p^2} - 1,$$

we see that for some a , T coincides with T_a . □

4 Large super-Vandermonde sets

Proposition 4.1 *Let T be an sV-set in $GF(q)$, $q = p^h$ of size $t > q/p$, then*

$$\prod_{y \in T} (Y - y^{-1}) = Y^t - c$$

for some t -th power $c \in GF(q)^$, so T is coset of a multiplicative subgroup.*

Proof As before $\phi(Y) = \prod_{y \in T} (Y - y^{-1}) = Y^t + g(Y)$, where g is a p -th power. Since this polynomial is fully reducible we may write:

$$(Y^t + g)(h_0 + Yh_1 + \cdots + Y^{p-1}h_{p-1}) = Y^q - Y, \quad q = p^h,$$

where also the polynomials h_i are p -th powers. We now equate left and right the terms of degree $d \bmod p$, $d = 0, p-1, \dots, 1$, writing $e = t - q/p$ and $E = q/p$:

$$\begin{aligned} gh_0 + Y^E h_{p-e} Y^p &= Y^q \\ gh_{p-1} + Y^E h_{p-e-1} &= 0 \\ &\vdots \\ gh_{e+1} + Y^E h_1 &= 0 \\ gh_e + Y^E h_0 &= 0 \\ gh_{e-1} + Y^E h_{p-1} Y^p &= 0 \\ gh_{e-2} + Y^E h_{p-2} Y^p &= 0 \\ &\vdots \\ gh_2 + Y^E h_{p-e+2} Y^p &= 0 \\ gh_1 + Y^E h_{p-e+1} Y^p &= -1 \end{aligned}$$

We look at the divisibility by Y . From the last equation we see that h_1 is not divisible by Y , in particular $h_1 \neq 0$, then we see from the other equation involving h_1 that h_{1+e} is divisible by Y^E , next h_{1+2e} by Y^{2E} , (where of course we take indices mod p) and so on until finally $h_{1+(p-1)e} = h_{p-e+1}$ is divisible by $Y^{(p-1)E} = Y^{q-q/p}$. If h_{p-e+1} is nonzero then the total degree of the left hand side will be at least $t + 1 + q - q/p > q$, a contradiction, so $h_{p-e+1} = 0$ and now the last equation tells us that g is constant. □

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